§4.1 #60 Find the absolute maximum and absolute minimum values of \( f(x) = x - \ln x \) on the interval \([\frac{1}{2}, 2]\).

Since \( f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x} \) we have \( f'(x) = 0 \) when \( x = 1 \) (note that 0 is not in the domain of \( f \)). Since \( f\left(\frac{1}{2}\right) = \frac{1}{2} - \ln \frac{1}{2} \approx 1.19 \), \( f(1) = 1 \) and \( f(2) = 2 - \ln 2 \approx 1.31 \) we see that \( f(2) = 2 - \ln 2 \) is the absolute maximum value and \( f(1) = 1 \) is the absolute minimum value on the interval \([\frac{1}{2}, 2]\).

§4.1 #66 (a) Use a graph to estimate the absolute maximum and minimum values of the function \( f(x) = e^x + e^{-2x} \) on the interval \( 0 \leq x \leq 1 \) to two decimal places. (b) Use calculus to find the exact maximum and minimum values.

(a) The absolute maximum value is about \( f(1) = 2.85 \), and the absolute minimum value is about \( f(0.23) = 1.89 \).

(b) Since \( f'(x) = e^x - 2e^{-2x} = e^{-2x}(e^{3x} - 2) \) we see \( f'(x) = 0 \) only at \( e^{3x} = 2 \), \( 3x = \ln 2 \). So we got \( x = \frac{1}{3} \ln 2 \approx 0.23 \). We then have \( f\left(\frac{1}{3} \ln 2\right) = (e^{\ln 2})^{1/3} + (e^{\ln 2})^{-2/3} = 2^{1/3} + 2^{-2/3} \approx 1.89 \) (minimum) and \( f(1) = e^1 + e^{-2} \approx 2.85 \) (maximum). Note that at the other endpoint, which should always be checked, \( f(0) = 2 \) is neither a max nor a min.

§4.2 #12 Verify that the function satisfies the hypothesis of the mean value Theorem on the given interval. Then find all numbers \( c \) that satisfy the conclusion of the mean value Theorem.

The function \( f(x) = \frac{1}{x} \) is continuous on \([1, 3]\) and differentiable on \((1, 3)\) since \( f'(x) = -\frac{1}{x^2} \) and so the MVT applies. Since \( \frac{f(3) - f(1)}{3-1} = -1/3 \), we solve \( f'(c) = -1/3 \) to get \( c = \pm \sqrt{3} \), but only use \( c = \sqrt{3} \) since \( c \) must lie in the interval \((1, 3)\).
§4.2 #17 Show that the equation $2x + \cos x = 0$ has exactly one real root.

Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$. Since $f$ is the sum of the polynomial $2x$ and the function $\cos x$, $f$ is continuous and differentiable for all $x$. By the IVT, there is a number $c$ in $(-\pi, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real root.

To show that there cannot be two distinct real roots, we investigate what would happen if there were distinct real roots $a$ and $b$ with $a < b$ giving $f(a) = f(b) = 0$. Since $f$ is continuous on $[a, b]$ and differentiable on $(a,b)$, Rolle’s Theorem implies that there is a number $r$ in $(a,b)$ such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \leq 1$. This contradiction shows that the given equation can’t have two distinct real roots, so it has exactly one root.

§4.3 #6 The graph of the derivative $f'$ is shown (in the text). (a) On what intervals is $f$ increasing or decreasing? (b) At what values of $x$ does $f$ have a local maximum or minimum?

(a) Since $f'(x) > 0$, $f$ is increasing on $(0,1)$ and $(3,5)$. Since $f'(x) < 0$, $f$ is decreasing on $(1,3)$ and $(5,6)$.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$ and $f'$ changes from positive to negative at both values, $f$ changes from increasing to decreasing and has local maxima at each of $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$ and $f'$ changes from negative to positive there, $f$ changes from decreasing to increasing and has a local minimum at $x = 3$. 
§4.3 #22 (a) Find the critical numbers of \( f(x) = x^4 \cdot (x - 1)^3 \). (b) What does the Second Derivative Test tell you about the behavior of \( f \) at these critical numbers? (c) What does the first derivative test tell you?

a) \( f'(x) = 4x^3(x-1)^3 + x^4(3)(x-1)^2 = x^3(x-1)^2[4(x-1)+3x] = x^3(x-1)^2(7x-4) \)

and so \( f'(x) = 0 \) when \( x = 0, 1, 4/7 \).

b) \( f''(x) = 3x^2(x - 1)^2(7x - 4) + x^3(2)(x - 1)(7x - 4) + x^3(x - 1)^2(7) \). Since \( f''(0) = 0 \) the second derivative test gives us no information about \( x = 0 \) (that is, it could be a local max or local min or neither at this point). Also, since \( f''(1) = 0 \) the second derivative test gives us no information about \( x = 1 \). But, since \( f''(4/7) > 0 \) we see that \( f \) has a local min at \( x = 4/7 \).

c) The first derivative test shows us that \( f' > 0 \) (and thus \( f \) is increasing) on the intervals \((-\infty, 0), (4/7, 1) \) and \((1, +\infty)\), while \( f' < 0 \) (and thus \( f \) is decreasing) on the interval \((0, 4/7)\). Hence the first derivative test tells us that \( f \) has a local max at \( x = 0 \), a local min at \( x = 4/7 \) and neither at \( x = 1 \).

§4.3 #41 (a) Find the intervals of increase or decrease. (b) Find the local maximum and minimum values. (c) Find the intervals of concavity and the inflection points. (d) Use the information from parts (a)-(c) to sketch the graph. Check your work with a graphing device if you have one.

Since \( C'(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \), we have \( C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4(x+1)}{3\sqrt[3]{x^2}} \). Hence we have critical points at \( x = -1, 0 \). Thus we see \( C'(x) > 0 \) if \(-1 < x < 0 \) or \( x > 0 \) and \( C'(x) < 0 \) for \( x < -1 \), so \( C \) is increasing on \((-1, \infty)\), and \( C \) is decreasing on the interval \((-\infty, -1)\). Hence \( C \) has a local min \( C(-1) = -3 \) attained at \( x = -1 \).

Since \( C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4(x-2)}{9\sqrt[3]{x^5}} \). \( C''(x) < 0 \) for \( 0 < x < 2 \) and \( C''(x) > 0 \) for \( x < 0 \) and \( x > 2 \), so \( C \) is concave downward on \((0, 2)\) and concave upward on \((-\infty, 0)\) and \((2, +\infty)\). There are inflection points at \((0, 0)\) and \((2, 6\sqrt[3]{2}) \approx (2, 7.56)\).

Use your calculator or Mathematica to see the graph.